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Construction of Nearest Points in the L^p, p even, and L^∞ norms. I*

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1. INTRODUCTION

It has long been of interest to find methods for constructing *nearest points* or *best approximations* in normed linear spaces. More precisely, if X is a normed linear space and M is a closed convex subset, then one seeks methods for constructing, for each f, points g^* in M such that $\| f - g^* \| \leqslant \| f - g \|$, for all $g \in M$.

We are concerned here with the space $C_{L^p}(T)$, $p = 2m$, $2 \leqslant m \leqslant \infty$, i.e., the space of all real valued continuous functions, defined on the compact measurable subset T of n space, endowed with the L^p norm. Furthermore, we choose M to be a *finite-dimensional linear subspace*. In Section 2 we introduce a new method for constructing nearest points in case m is finite. Using the results of Section 2, we give, in Section 3, a method for constructing nearest points in the L^∞ norm. This method differs significantly from the Pólya algorithm [1] by not requiring the L^p approximations, p finite, to be actually computed. A subsequent article will treat applications, complex valued functions, and some subsets M, which are not linear subspaces.

Throughout the paper, we restrict ourselves to the situation where the linear subspace M and the point f satisfy the relation

$$f(x) \neq g(x), \qquad \text{a.e., for all } g \in M. \tag{1}$$

We shall make use of the notation

$$\| h \|_{L^2,w} = \left[\int_T h^2(t)\, w(t)\, dt \right]^{1/2}.$$

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2. L^p Norms, $p = 2m$, $2 \leqslant m < \infty$

For such a p, and M and f satisfying (1), we construct inductively the sequence of approximations

$$g_0, h_0, g_1, h_1, \ldots, g_n, h_n, \ldots \qquad (g_0 \text{ given}), \tag{2}$$

as follows. Given g_n, then $h_n \in M$ is uniquely defined by

$$\| f - h_n \|_{L^2,w} \leqslant \| f - g \|_{L^2,w}, \qquad \text{for all } g \in M, \tag{3}$$

where $w = (f - g_n)^{p-2}$. Given h_n, then g_{n+1} is uniquely defined by

$$g_{n+1} = g_n + \lambda_n(h_n - g_n), \tag{4}$$

where λ_n minimizes $\| f - g_n - \lambda(h_n - g_n)\|_{L^p}$ over all real λ. We note that (3) is equivalent to

$$\int_T [f(t) - g_n(t)]^{p-2}[f(t) - h_n(t)]\, g(t)\, dt = 0, \qquad \text{for all } g \in M. \tag{5}$$

Hence, h_n can be found by solving a finite number of linear equations. We also note that g_{n+1} is found by minimizing the convex function $\phi(\lambda) = \| f - g_n - \lambda(h_n - g_n)\|_{L^p}$, $-\infty < \lambda < +\infty$.

The unique point in M which lies nearest to f in the L^p norm, p finite, is denoted by G_p; thus,

$$\| f - G_p \|_{L^p} \leqslant \| f - g \|_{L^p}, \qquad \text{for all } g \in M.$$

The following theorem holds.

Theorem 1. *Let $p = 2m$, $2 \leqslant m < \infty$, and let f and M satisfy* (1). *Then for any initial approximation $g_0 \in M$, the sequence* (2) *satisfies the following*:

1. *For each n, either $g_n = G_p$ or*

$$\| f - g_{n+1} \|_{L^p} < \| f - g_n \|_{L^p};$$

2. *$h_n, g_n \to G_p$, uniformly, as $n \to \infty$;*

3. *for each n, either $g_n = G_p$ or*

$$\| f - h_n \|_{L^2,w} \leqslant \| f - G_p \|_{L^2,w} < \| f - g_n \|_{L^2,w},$$

where $w = (f - g_n)^{p-2}$.

Proof. *Part* 1. By construction, $\|f - g_{n+1}\|_{L^p} \leqslant \|f - g_n\|_{L^p}$. If equality holds then by differentiation we find

$$\int_T (f(t) - g_n(t))^{p-1}(h_n(t) - g_n(t))\, dt = 0.$$

Combining this with (5), we derive

$$\int_T (f(t) - g_n(t))^p\, dt = \int_T (f(t) - g_n(t))^{p-2}(f(t) - h_n(t))^2\, dt.$$

By the uniqueness of h_n, this implies that $h_n = g_n$. Substituting in (5), we find

$$\int_T (f(t) - g_n(t))^{p-1} g(t)\, dt = 0, \qquad \text{for all } g \in M,$$

which implies that $g_n = G_p$; thus, Part 1 is completed .

Part 2. It can be shown that the sequence (2) is bounded. By virtue of this and the uniqueness of G_p, it is sufficient to show that if subsequences $\{g_{n_k}\}$ and $\{h_{n_k}\}$ converge uniformly to g^* and h^*, respectively, then $g^* = h^* = G_p$. To this end, we note that from (5) and the uniform convergence we have

$$\int_T [f(t) - g^*(t)]^{p-2}[f(t) - h^*(t)]\, g(t)\, dt = 0, \qquad \text{for all } g \in M. \tag{6}$$

Now suppose that $\|f - g^* - \lambda^*(h^* - g^*)\|_{L^p} < \|f - g^*\|_{L^p}$, for some λ^*. Then, by uniform convergence, $\|f - g_{n_k} - \lambda^*(h_{n_k} - g_{n_k})\|_{L^p} < \|f - g^*\|_{L^p}$, for some k. By definition, $\|f - g_{n_k+1}\|_{L^p} \leqslant \|f - g_{n_k} - \lambda^*(h_{n_k} - g_{n_k})\|_{L^p}$, and by Part 1 above, $\|f - g^*\|_{L^p} \leqslant \|f - g_{n_{k+1}}\|_{L^p} \leqslant \|f - g_{n_k+1}\|_{L^p}$. Thus, we arrive at the contradiction: $\|f - g_{n_k+1}\|_{L^p} < \|f - g_{n_k+1}\|_{L^p}$. Hence,

$$\|f - g^*\|_{L^p} \leqslant \|f - g^* - \lambda(h^* - g^*)\|_{L^p}, \qquad \text{for all } \lambda. \tag{7}$$

By virtue of (6) and (7), we can apply the proof of Part 1 to conclude that $g^* = h^* = G_p$. This proves Part 2.

Part 3. If $g_n \neq G_p$, then the inequality $\|f - h_n\|_{L^2,w} \leqslant \|f - G_p\|_{L^2,w}$, where $w = (f - g_n)^{p-2}$, follows from the definition of h_n. The second inequality $\|f - G_p\|_{L^2,w} < \|f - g_n\|_{L^2,w}$ follows by a direct application of Hölder's inequality. This completes the proof.

3. L^{∞} Norm

Given f and M satisfying (1), we construct inductively the sequence of approximations

$$H_2, H_4, \ldots, H_{2m}, \ldots \qquad (H_2 \text{ given}), \tag{8}$$

as follows. Given $H_{2(m-1)}$, H_{2m} is found by constructing the finite sequence

$$H_{2(m-1)} = g_0, h_0, \ldots, g_n, h_n, \ldots, g_{k(m)}, h_{k(m)} = H_{2m},$$

according to (3) and (4), above, with $p = 2m$. The integer $k(m)$ is chosen so that

$$\| f - g_{k(m)} \|^2_{L^2,w} - \| f - h_{k(m)} \|^2_{L^2,w} \leqslant 1/2^{m^2}, \tag{9}$$

where $w = (f - g_{k(m)})^{2m-2}$. According to Theorem 1, such an integer exists. We denote by M_{∞} the set of all points in M which lie nearest to f in the L^{∞} norm; thus,

$$M_{\infty} = \{G_{\infty} : G_{\infty} \in M, \| f - G_{\infty} \|_{L^{\infty}} \leqslant \| f - g \|_{L^{\infty}}, \text{ for all } g \in M\}.$$

We now have

Theorem 2. *Let f and M satisfy* (1). *If M_{∞} consists of only one point G_{∞}, then, for any initial approximation $H_2 \in M$, the sequence* (8) *converges uniformly to G_{∞}. If M_{∞} consists of more than one point, then the sequence* (8) *is bounded and every convergent subsequence of it converges uniformly to some point of M_{∞}.*

Proof. From inequality (9) and Part 3 of Theorem 1, it follows that

$$\| f - G_{2m} \|^2_{L^2,w} - \| f - H_{2m} \|^2_{L^2,w} \leqslant 1/2^{m^2},$$

where $w = (f - g_{k(m)})^{2m-2}$. Employing the orthogonality property (5) of H_{2m}, this implies

$$\| H_{2m} - G_{2m} \|^2_{L^2,w} \leqslant 1/2^{m^2}, \tag{10}$$

where $w = [f - g_{k(m)}]^{2m-2}$. It is well known (Pólya [1], see also Buck [2]) that if M_{∞} consists of only one point G_{∞}, then $G_{2m} \to G_{\infty}$. By virtue of relations (1) and (10), it is readily shown that $H_{2m} \to G_{\infty}$, uniformly. If M_{∞} consists of more than one point, it is known that $\{G_{2m}\}$ is bounded and every convergent subsequence of it converges to some point of M_{∞}. Again using relations (1) and (10), the desired properties of $\{H_{2m}\}$ follow readily.

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